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# On the Uniqueness of Pairs of a Hamiltonian and a Strong Time Operator in Quantum Mechanics

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## Abstract

Let  $H$  be a self-adjoint operator (a Hamiltonian) on a complex Hilbert space  $\mathcal{H}$ . A symmetric operator  $T$  on  $\mathcal{H}$  is called a *strong time operator* of  $H$  if the pair  $(T, H)$  obeys the operator equation  $e^{itH}Te^{-itH} = T + t$  for all  $t \in \mathbb{R}$  ( $\mathbb{R}$  is the set of real numbers and  $i$  is the imaginary unit). In this note we review some results on the uniqueness (up to unitary equivalences) of the pairs  $(T, H)$ .

*Keywords:* canonical commutation relation, Hamiltonian, strong time operator, weak Weyl relation, weak Weyl representation, Weyl representation, spectrum.

Mathematics Subject Classification 2000: 81Q10, 47N50

## 1 Introduction

A pair  $(T, H)$  of a symmetric operator  $T$  and a self-adjoint operator  $H$  on a complex Hilbert space  $\mathcal{H}$  is called a *weak Weyl representation* of the canonical commutation relation (CCR) with one degree of freedom if it obeys the *weak Weyl relation*: For all  $t \in \mathbb{R}$  (the set of real numbers),  $e^{-itH}D(T) \subset D(T)$  ( $i$  is the imaginary unit and  $D(T)$  denotes the domain of  $T$ ) and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi, \quad \forall t \in \mathbb{R}, \forall \psi \in D(T). \quad (1.1)$$

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It is easy to see that the weak Weyl relation is equivalent to the *operator equation*

$$e^{itH}Te^{-itH} = T + t, \quad \forall t \in \mathbb{R}, \quad (1.2)$$

implying that  $e^{-itH}D(T) = D(T), \forall t \in \mathbb{R}$ .

One can prove that, if  $(T, H)$  is a weak Weyl representation of the CCR, then  $(T, H)$  obeys the CCR

$$[T, H] = i \quad (1.3)$$

on  $D(TH) \cap D(HT)$ , where  $[X, Y] := XY - YX$ . But *the converse is not true*.

In the context of quantum theory where  $H$  is the Hamiltonian of a quantum system,  $T$  is called a *strong time operator* of  $H$  [3, 5].

We remark that a standard time operator (simply a time operator) of  $H$  is defined to be a symmetric operator  $T$  on  $\mathcal{H}$  obeying CCR (1.3) on a subspace  $\mathcal{D} \neq \{0\}$  (not necessarily dense) of  $\mathcal{H}$  (i.e.,  $\mathcal{D} \subset D(TH) \cap D(HT)$  and  $[T, H]\psi = i\psi, \forall \psi \in \mathcal{D}$ ) (cf. [1]). Obviously this notion of time operator is weaker than that of strong time operator. General classes of time operators (not strong ones) of a Hamiltonian with discrete eigenvalues have been investigated by Galapon [12], Arai-Matsuzawa [9] and Arai [7].

Weak Weyl representations of the CCR were first discussed by Schmüdgen [19, 20] from a purely operator theoretical point of view and then by Miyamoto [14] in application to a theory of time operator in quantum theory. A generalization of a weak Weyl relation was presented by the present author [2] to cover a wider range of applications to quantum physics including quantum field theory.

Arai-Matsuzawa [8] discovered a general structure for construction of a weak Weyl representation of the CCR from a given weak Weyl representation and established a theorem for the former representation to be a Weyl representation of the CCR. These results were extended by Hiroshima-Kuribayashi-Matsuzawa [13] to a wider class of Hamiltonians.

In the previous paper [6] the author considered the problem on uniqueness (up to unitary equivalences) of weak Weyl representations. In the context of theory of time operators, this is a problem on uniqueness (up to unitary equivalences) of pairs  $(T, H)$  with  $H$  a Hamiltonian and  $T$  a strong time operator of  $H$ . This problem has an independent interest in the theory of weak Weyl representations. This note is a review of some results obtained in [6].

## 2 Preliminaries

We denote by  $W(\mathcal{H})$  the set of all the weak Weyl representations on  $\mathcal{H}$ :

$$W(\mathcal{H}) := \{(T, H) | (T, H) \text{ is a weak Weyl representation on } \mathcal{H}\}. \quad (2.1)$$

It is easy to see that, if  $(T, H)$  is in  $W(\mathcal{H})$ , then so are  $(\overline{T}, H)$  and  $(-T, -H)$ , where  $\overline{T}$  denotes the closure of  $T$ .

For a linear operator  $A$  on a Hilbert space,  $\sigma(A)$  (resp.  $\rho(A)$ ) denotes the spectrum (resp. the resolvent set) of  $A$  (if  $A$  is closable, then  $\sigma(A) = \sigma(\overline{A})$ ). Let  $\mathbb{C}$  be the set of complex numbers and

$$\Pi_+ := \{z \in \mathbb{C} | \operatorname{Im} z > 0\}, \quad \Pi_- := \{z \in \mathbb{C} | \operatorname{Im} z < 0\}. \quad (2.2)$$

In the previous paper [4], we proved the following facts:

**Theorem 2.1** [4] *Let  $(T, H) \in W(\mathcal{H})$ . Then:*

- (i) *If  $H$  is bounded below, then either  $\sigma(T) = \overline{\Pi}_+$  (the closure of  $\Pi_+$ ) or  $\sigma(T) = \mathbb{C}$ .*
- (ii) *If  $H$  is bounded above, then either  $\sigma(T) = \overline{\Pi}_-$  or  $\sigma(T) = \mathbb{C}$ .*
- (iii) *If  $H$  is bounded, then  $\sigma(T) = \mathbb{C}$ .*

This theorem has to be taken into account in considering the uniqueness problem of weak Weyl representations.

A form of representations of the CCR stronger than weak Weyl representations is known as a *Weyl representation* of the CCR which is a pair  $(T, H)$  of *self-adjoint* operators on  $\mathcal{H}$  obeying the *Weyl relation*

$$e^{itT} e^{isH} = e^{-its} e^{isH} e^{itT}, \quad \forall t, \forall s \in \mathbb{R}. \quad (2.3)$$

It is well known (the von Neumann uniqueness theorem [15]) that, every Weyl representation on a *separable* Hilbert space is unitarily equivalent to a direct sum of the Schrödinger representation  $(q, p)$  on  $L^2(\mathbb{R})$ , where  $q$  is the multiplication operator by the variable  $x \in \mathbb{R}$  and  $p = -iD_x$  with  $D_x$  being the generalized differential operator in  $x$  (cf. [3, §3.5], [16, Theorem 4.3.1], [17, Theorem VIII.14]).

It is easy to see that a Weyl representation is a weak Weyl representation (but the converse is not true). Therefore, as far as the Hilbert space under consideration is separable, the non-trivial case for the uniqueness problem of weak Weyl representations is the one where they are *not* Weyl representations. A general class of such weak Weyl representations  $(T, H)$  are given in the case where  $H$  is semi-bounded (bounded below or bounded above). In this case,  $T$  is not essentially self-adjoint [2, Theorem 2.8], implying Theorem 2.1.

Two simple examples in this class are constructed as follows:

**Example 2.1** Let  $a \in \mathbb{R}$  and consider the Hilbert space  $L^2(\mathbb{R}_a^+)$  with  $\mathbb{R}_a^+ := (a, \infty)$ . Let  $q_{a,+}$  be the multiplication operator on  $L^2(\mathbb{R}_a^+)$  by the variable  $\lambda \in \mathbb{R}_a^+$ :

$$D(q_{a,+}) := \left\{ f \in L^2(\mathbb{R}_a^+) \mid \int_a^\infty \lambda^2 |f(\lambda)|^2 d\lambda < \infty \right\}, \quad (2.4)$$

$$q_{a,+}f := \lambda f, \quad f \in D(q_{a,+}) \quad (2.5)$$

and

$$p_{a,+} := -i \frac{d}{d\lambda} \quad (2.6)$$

with  $D(p_{a,+}) = C_0^\infty(\mathbb{R}_a^+)$ , the set of infinitely differentiable functions on  $\mathbb{R}_a^+$  with bounded support in  $\mathbb{R}_a^+$ . Then it is easy to see that  $q_{a,+}$  is self-adjoint, bounded below with  $\sigma(q_{a,+}) = [a, \infty)$  and  $p_{a,+}$  is a symmetric operator. Moreover,  $(-p_{a,+}, q_{a,+})$  is a weak Weyl representation of the CCR. Hence, as remarked above,  $(-\bar{p}_{a,+}, q_{a,+})$  also is a weak Weyl representation.

Note that  $p_{a,+}$  is not essentially self-adjoint and

$$\sigma(-p_{a,+}) = \sigma(-\bar{p}_{a,+}) = \bar{\Pi}_+. \quad (2.7)$$

In particular,  $\pm \bar{p}_{a,+}$  are maximal symmetric, i.e., they have no non-trivial symmetric extensions (e.g., [18, §X.1, Corollary]).

**Example 2.2** Let  $b \in \mathbb{R}$  and consider the Hilbert space  $L^2(\mathbb{R}_b^-)$  with  $\mathbb{R}_b^- := (-\infty, b)$ . Let  $q_{b,-}$  be the multiplication operator on  $L^2(\mathbb{R}_b^-)$  by the variable  $\lambda \in \mathbb{R}_b^-$ . and

$$p_{b,-} := -i \frac{d}{d\lambda} \quad (2.8)$$

with  $D(p_{b,-}) = C_0^\infty(\mathbb{R}_b^-)$ . Then  $q_{b,-}$  is self-adjoint, bounded above with  $\sigma(q_{b,-}) = (-\infty, b]$ ,  $p_{b,-}$  is a symmetric operator, and  $(-p_{b,-}, q_{b,-})$  is a weak Weyl representation of the CCR. As in the case of  $p_{a,+}$ ,  $p_{b,-}$  is not essentially self-adjoint and

$$\sigma(-p_{b,-}) = \bar{\Pi}_-. \quad (2.9)$$

A relation between  $(-p_{a,+}, q_{a,+})$  and  $(-p_{b,-}, q_{b,-})$  is given as follows. Let  $U_{ab} : L^2(\mathbb{R}_a^+) \rightarrow L^2(\mathbb{R}_b^-)$  be a linear operator defined by

$$(U_{ab}f)(\lambda) := f(a + b - \lambda), \quad f \in L^2(\mathbb{R}_a^+), \text{ a.e. } \lambda \in \mathbb{R}_b^-.$$

Then  $U_{ab}$  is unitary and

$$U_{ab}q_{a,+}U_{ab}^{-1} = a + b - q_{b,-}, \quad U_{ab}p_{a,+}U_{ab}^{-1} = -p_{b,-}. \quad (2.10)$$

In view of the von Neumann uniqueness theorem for Weyl representations, the pair  $(-\bar{p}_{a,+}, q_{a,+})$  (resp.  $(-\bar{p}_{b,-}, q_{b,-})$ ) may be a reference pair in classifying weak Weyl representations  $(T, H)$  with  $H$  being bounded below (resp. bounded above).

By Theorem 2.1, we can define two subsets of  $W(\mathcal{H})$ :

$$W_+(\mathcal{H}) := \{(T, H) \in W(\mathcal{H}) | H \text{ is bounded below and } \sigma(T) = \bar{\Pi}_+\}, \quad (2.11)$$

$$W_-(\mathcal{H}) := \{(T, H) \in W(\mathcal{H}) | H \text{ is bounded above and } \sigma(T) = \bar{\Pi}_-\}. \quad (2.12)$$

Then, as shown above,  $(-p_{a,+}, q_{a,+}) \in W_+(L^2(\mathbb{R}_a^+))$  and  $(-p_{b,-}, q_{b,-}) \in W_-(L^2(\mathbb{R}_b^-))$ .

### 3 Irreducibility

For a set  $\mathcal{A}$  of linear operators on a Hilbert space  $\mathcal{H}$ , we set

$$\mathcal{A}' := \{B \in \mathbf{B}(\mathcal{H}) | BA \subset AB, \forall A \in \mathcal{A}\},$$

called the *strong commutant* of  $\mathcal{A}$  in  $\mathcal{H}$ , where  $\mathbf{B}(\mathcal{H})$  is the set of all bounded linear operators on  $\mathcal{H}$  with  $D(B) = \mathcal{H}$ .

We say that  $\mathcal{A}$  is *irreducible* if  $\mathcal{A}' = \{cI | c \in \mathbb{C}\}$ , where  $I$  is the identity on  $\mathcal{H}$ .

**Proposition 3.1** *For all  $a \in \mathbb{R}$ , the set  $\{\bar{p}_{a,+}, p_{a,+}^*, q_{a,+}\}$  (Example 2.1) is irreducible.*

To prove this proposition, we need a lemma.

Let  $a \in \mathbb{R}$  be fixed. For each  $t \geq 0$ , we define a linear operator  $U_a(t)$  on  $L^2(\mathbb{R}_a^+)$  as follows: For each  $f \in L^2(\mathbb{R}_a^+)$ ,

$$(U_a(t)f)(\lambda) := \begin{cases} f(\lambda - t) & \lambda > t + a \\ 0 & a < \lambda \leq t + a \end{cases} \quad (3.1)$$

Then it is easy to see that  $\{U_a(t)\}_{t \geq 0}$  is a strongly continuous one-parameter semi-group of isometries on  $L^2(\mathbb{R}_a^+)$ .

**Lemma 3.2** *The generator of  $\{U_a(t)\}_{t \geq 0}$  is  $-i\bar{p}_{a,+}$ :*

$$\frac{dU_a(t)f}{dt} = -i\bar{p}_{a,+}U_a(t)f, \quad \forall f \in D(\bar{p}_{a,+}), t \in \mathbb{R}, \quad (3.2)$$

where the derivative in  $t$  is taken in the strong sense.

*Proof.* Let  $iA$  be the generator of  $\{U_a(t)\}_{t \geq 0}$ :

$$\frac{dU_a(t)f}{dt} = iAU_a(t)f, \quad \forall f \in D(A), t \in \mathbb{R}.$$

Then it follows from the isometry of  $U_a(t)$  that  $A$  is a closed symmetric operator. It is easy to see that  $-p_{a,+} \subset A$  and hence  $-\bar{p}_{a,+} \subset A$ . As already remarked in Example 2.1,  $-\bar{p}_{a,+}$  is maximal symmetric. Hence  $A = -\bar{p}_{a,+}$ .  $\blacksquare$

## Proof of Proposition 3.1

Let  $B \in \{\bar{p}_{a,+}, p_{a,+}^*, q_{a,+}\}'$ . Then

$$B\bar{p}_{a,+} \subset \bar{p}_{a,+}B, \quad (3.3)$$

$$Bp_{a,+}^* \subset p_{a,+}^*B, \quad (3.4)$$

$$Bq_{a,+} \subset q_{a,+}B. \quad (3.5)$$

As in the case of bounded linear operators on  $L^2(\mathbb{R})$  strongly commuting with  $q$  (the multiplication operator by the variable  $x \in \mathbb{R}$ ) [3, Lemma 3.13], (3.5) implies that there exists an essentially bounded function  $F$  on  $\mathbb{R}_a^+$  such that  $B = M_F$ , the multiplication operator by  $F$ .

Let  $f \in D(\bar{p}_{a,+})$  and  $g(t) := BU_a(t)f$ . Then, by Lemma 3.2,  $g$  is strongly differentiable in  $t \geq 0$  and

$$\frac{dg(t)}{dt} = B(-i\bar{p}_{a,+})U_a(t)f = -i\bar{p}_{a,+}g(t),$$

where we have used (3.3). Note that  $g(0) = Bf$ . Hence, by the uniqueness of solutions of the initial value problem on differential equation (3.2), we have  $g(t) = U_a(t)Bf$ . Therefore it follows that  $BU_a(t) = U_a(t)B, \forall t \geq 0$ . Hence  $FU_a(t)f = U_a(t)Ff, \forall f \in L^2(\mathbb{R}_a^+)$ , which implies that

$$F(\lambda)f(\lambda - t) = F(\lambda - t)f(\lambda - t), \quad \lambda > t + a.$$

Hence  $F(\lambda) = F(\lambda + t)$ , a.e.  $\lambda > 0, \forall t > 0$ . This means that  $F$  is equivalent to a constant function. Hence  $B = M_F = cI$  with some  $c \in \mathbb{C}$ .  $\blacksquare$

**Proposition 3.3** *For all  $b \in \mathbb{R}$ , the set  $\{\bar{p}_{b,-}, p_{b,-}^*, q_{b,-}\}$  (Example 2.2) is irreducible.*

*Proof.* Let  $B \in \{\bar{p}_{b,-}, p_{b,-}^*, q_{b,-}\}'$ . Then, by (2.10), the operator  $C := U_{ab}^{-1}BU_{ab}$  is in  $\{\bar{p}_{a,+}, p_{a,+}^*, q_{a,+}\}'$ . Hence, by Proposition 3.1,  $C = cI$  with some constant  $c \in \mathbb{C}$ . Thus  $B = cI$ .  $\blacksquare$

## 4 Uniqueness Theorem

One can prove the following theorem:

**Theorem 4.1** *Let  $\mathcal{H}$  be separable and  $(T, H) \in W_+(\mathcal{H})$  with  $\varepsilon_0 := \inf \sigma(H)$ . Suppose that  $\{\bar{T}, T^*, H\}$  is irreducible. Then there exists a unitary operator  $U : \mathcal{H} \rightarrow L^2(\mathbb{R}_{\varepsilon_0}^+)$  such that*

$$U\bar{T}U^{-1} = -\bar{p}_{\varepsilon_0,+}, \quad UHU^{-1} = q_{\varepsilon_0,+}. \quad (4.1)$$

In particular

$$\sigma(H) = [\varepsilon_0, \infty). \quad (4.2)$$

**Remark 4.1** It is known that, for every weak Weyl representation  $(T, H) \in W(\mathcal{H})$  ( $\mathcal{H}$  is not necessarily separable),  $H$  is purely absolutely continuous [14, 19].

We prove Theorem 4.1 in the next section. For the moment, we note a result which immediately follows from Theorem 4.1:

**Theorem 4.2** Let  $\mathcal{H}$  be separable and  $(T, H) \in W_-(\mathcal{H})$  with  $b := \sup \sigma(H)$ . Suppose that  $\{\bar{T}, T^*, H\}$  is irreducible. Then there exists a unitary operator  $V : \mathcal{H} \rightarrow L^2(\mathbb{R}_b^-)$  such that

$$V\bar{T}V^{-1} = -\bar{p}_{b,-}, \quad VHV^{-1} = q_{b,-}. \quad (4.3)$$

In particular

$$\sigma(H) = (-\infty, b]. \quad (4.4)$$

*Proof.* As remarked in Section 2,  $(-T, -H) \in W_+(\mathcal{H})$  with  $a := \inf \sigma(-H) = -b$  and  $\sigma(-T) = \bar{\Pi}_+$ . Hence, we can apply Theorem 4.1 to conclude that there exists a unitary operator  $U : \mathcal{H} \rightarrow L^2(\mathbb{R}_a^+)$  such that

$$U\bar{T}U^{-1} = \bar{p}_{a,+}, \quad UHU^{-1} = -q_{a,+}.$$

By Example 2.2, we have

$$U_{ab}\bar{p}_{a,+}U_{ab}^{-1} = -\bar{p}_{b,-}, \quad U_{ab}q_{a,+}U_{ab}^{-1} = -q_{b,-},$$

where we have used that  $a+b=0$ . Hence, putting  $V := U_{ab}U$ , we obtain the desired result. ■

**Remark 4.2** In view of Theorems 4.1 and 4.2, it would be interesting to know when  $\sigma(T) = \bar{\Pi}_+$  (resp.  $\bar{\Pi}_-$ ) for  $(T, H) \in W(\mathcal{H})$  with  $H$  bounded below (resp. above). Concerning this problem, we have the following results [5]:

- (i) Let  $(T, H) \in W(\mathcal{H})$  and  $H$  be bounded below. Suppose that, for some  $\beta_0 > 0$ ,  $\text{Ran}(e^{-\beta_0 H} T)$  (the range of  $e^{-\beta_0 H} T$ ) is dense in  $\mathcal{H}$ . Then  $\sigma(T) = \bar{\Pi}_+$ .
- (ii) Let  $(T, H) \in W(\mathcal{H})$  and  $H$  be bounded above. Suppose that, for some  $\beta_0 > 0$ ,  $\text{Ran}(e^{\beta_0 H} T)$  is dense in  $\mathcal{H}$ . Then  $\sigma(T) = \bar{\Pi}_-$ .



## 5 Proof of Theorem 4.1

**Lemma 5.1** *Let  $S$  be a closed symmetric operator on  $\mathcal{H}$  such that  $\sigma(S) = \overline{\Pi}_+$ . Then there exists a unique strongly continuous one-parameter semi-group  $\{Z(t)\}_{t \geq 0}$  whose generator is  $iS$ . Moreover, each  $Z(t)$  is an isometry:*

$$Z(t)^*Z(t) = I, \quad \forall t \geq 0. \quad (5.1)$$

*Proof.* This fact is probably well known. But, for completeness, we give a proof. By the assumption  $\sigma(S) = \overline{\Pi}_+$ , we have  $\sigma(iS) = \{z \in \mathbb{C} | \operatorname{Re} z \leq 0\}$ . Therefore the positive real axis  $(0, \infty)$  is included in the resolvent set  $\rho(iS)$  of  $iS$ . Since  $S$  is symmetric, it follows that

$$\|(iS - \lambda)^{-1}\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Hence, by the Hille-Yosida theorem,  $iS$  generates a strongly continuous one-parameter semi-group  $\{Z(t)\}_{t \geq 0}$  of contractions. For all  $\psi \in D(iS) = D(S)$ ,  $Z(t)\psi$  is in  $D(S)$  and strongly differentiable in  $t \geq 0$  with

$$\frac{d}{dt}Z(t)\psi = iSZ(t)\psi = Z(t)iS\psi.$$

This equation and the symmetricity of  $S$  imply that  $\|Z(t)\psi\|^2 = \|\psi\|^2, \forall t \geq 0$ . Hence (5.1) follows.  $\blacksquare$

**Lemma 5.2** *Let  $(T, H) \in W_+(\mathcal{H})$ . Then there exists a unique strongly continuous one-parameter semi-group  $\{U_T(t)\}_{t \geq 0}$  whose generator is  $i\overline{T}$ . Moreover, each  $U_T(t)$  is an isometry and*

$$U_T(t)e^{-isH} = e^{its}e^{-isH}U_T(t), \quad t \geq 0, s \in \mathbb{R}. \quad (5.2)$$

*Proof.* We can apply Lemma 5.1 to  $S = \overline{T}$  to conclude that  $i\overline{T}$  generates a strongly continuous one-parameter semi-group  $\{U_T(t)\}_{t \geq 0}$  of isometries on  $\mathcal{H}$ . For all  $\psi \in D(\overline{T})$  and all  $t \geq 0$ ,  $U_T(t)\psi$  is in  $D(\overline{T})$  and strongly differentiable in  $t \geq 0$  with

$$\frac{d}{dt}U_T(t)\psi = i\overline{T}U_T(t)\psi = U_T(t)i\overline{T}\psi.$$

Let  $s \in \mathbb{R}$  be fixed and  $V(t) := e^{its}e^{-isH}U_T(t)e^{isH}$ . Then  $\{V(t)\}_{t \geq 0}$  is a strongly continuous one-parameter semi-group of isometries. Let  $\psi \in D(\overline{T})$ . Then  $e^{-isH}\psi \in D(\overline{T})$  and

$$\overline{T}e^{-isH}\psi = e^{-isH}\overline{T}\psi + se^{-isH}\psi.$$

Hence  $V(t)\psi$  is in  $D(\overline{T})$  and strongly differentiable in  $t$  with

$$\frac{d}{dt}V(t)\psi = i\overline{T}V(t)\psi.$$

This implies that  $V(t)\psi = U_T(t)\psi, \forall t \in \mathbb{R}$ . Since  $D(\overline{T})$  is dense, it follows that  $V(t) = U_T(t), \forall t \in \mathbb{R}$ , implying (5.2).  $\blacksquare$

We recall a result of Bracci and Picasso [10]. Let  $\{U(\alpha)\}_{\alpha \geq 0}$  and  $\{V(\beta)\}_{\beta \in \mathbb{R}}$  be a strongly continuous one-parameter semi-group and a strongly continuous one-parameter unitary group on  $\mathcal{H}$  respectively, satisfying

$$U(\alpha)^*U(\alpha) = I, \quad \alpha \geq 0, \quad (5.3)$$

$$U(\alpha)V(\beta) = e^{i\alpha\beta}V(\beta)U(\alpha), \quad \alpha \geq 0, \beta \in \mathbb{R}. \quad (5.4)$$

Then, by the Stone theorem, there exists a unique self-adjoint operator  $P$  on  $\mathcal{H}$  such that

$$V(\beta) = e^{-i\beta P}, \quad \beta \in \mathbb{R}. \quad (5.5)$$

**Lemma 5.3** [10] *Let  $\mathcal{H}$  be separable and  $P$  is bounded below with  $\nu := \inf \sigma(P)$ . Suppose that  $\{U(\alpha), U(\alpha)^*, V(\beta) | \alpha \geq 0, \beta \in \mathbb{R}\}$  is irreducible. Then, there exists a unitary operator  $Y : \mathcal{H} \rightarrow L^2(\mathbb{R}_+^+)$  such that*

$$YV(\beta)Y^{-1} = e^{-i\beta q_{\nu,+}}, \beta \in \mathbb{R}, \quad (5.6)$$

$$YU(\alpha)Y^{-1} = U_{\nu}(\alpha), \quad \alpha \geq 0. \quad (5.7)$$

We denote the generator of  $\{U(\alpha)\}_{\alpha \geq 0}$  by  $iQ$ . It follows that  $Q$  is closed and symmetric.

**Lemma 5.4** *Under the assumption of Lemma 5.3,*

$$YPY^{-1} = q_{\nu,+}, \quad (5.8)$$

$$YQY^{-1} = -\overline{p}_{\nu,+}. \quad (5.9)$$

*In particular*

$$\sigma(P) = [\nu, \infty). \quad (5.10)$$

*Proof.* Lemma 5.3 and (5.6) imply (5.8). Similarly (5.9) follows from Lemma 5.3, (5.7) and Lemma 3.2.  $\blacksquare$

**Lemma 5.5** *Let  $(T, H) \in W(\mathcal{H})$  with  $\sigma(T) = \overline{\Pi}_+$ . Suppose that  $\{\overline{T}, T^*, H\}$  is irreducible. Then  $\{U_T(t), U_T(t)^*, e^{-isH} | t \geq 0, s \in \mathbb{R}\}$  is irreducible.*

*Proof.* Let  $B \in \mathcal{B}(\mathcal{H})$  be such that

$$BU_T(t) = U_T(t)B, \quad (5.11)$$

$$BU_T(t)^* = U_T(t)^*B, \quad (5.12)$$

$$Be^{-isH} = e^{-isH}B, \forall t \geq 0, \forall s \in \mathbb{R}. \quad (5.13)$$

Let  $\psi \in D(\bar{T})$ . Then, by (5.11), we have  $BU_T(t)\psi = U_T(t)B\psi, \forall t \geq 0$ . By Lemma 5.2, the left hand side is strongly differentiable in  $t$  with  $d(BU_T(t)\psi)/dt = iB\bar{T}U_T(t)\psi$ . Hence so does the right hand side and we obtain that  $B\psi \in D(\bar{T})$  and  $B\bar{T}\psi = \bar{T}B\psi$ . Therefore  $B\bar{T} \subset \bar{T}B$ . Note that (5.12) implies that  $U_T(t)B^* = B^*U_T(t)$ . Hence it follows that  $B^*\bar{T} \subset \bar{T}B^*$ , which implies that  $BT^* \subset T^*B$ , where we have used the following general facts: for every densely defined closable linear operator  $A$  on  $\mathcal{H}$  and all  $C \in \mathcal{B}(\mathcal{H})$ ,  $(CA)^* = A^*C^*$ ,  $(AC)^* \supset C^*A^*$ ,  $(\bar{A})^* = A^*$ . Similarly (5.13) implies that  $BH \subset HB$ . Hence  $B \in \{\bar{T}, T^*, H\}'$ . Therefore  $B = cI$  for some  $c \in \mathbb{C}$ .

## Proof of Theorem 4.1

By Lemmas 5.2 and 5.5, we can apply Lemma 5.3 to the case where  $V(\beta) = e^{-i\beta H}, \beta \in \mathbb{R}$  and  $U(\alpha) = U_T(\alpha), \alpha \geq 0$ . Then the desired results follow from Lemmas 5.3 and 5.4.  $\blacksquare$

**Remark 5.1** Recently Bracci and Picasso [11] have obtained an interesting result on the reducibility of the von Neumann algebra generated by  $\{U(\alpha), U(\alpha)^*, V(\beta) | \alpha \geq 0, \beta \in \mathbb{R}\}$  obeying (5.3) and (5.4). By employing the result, one can generalize Theorem 4.1 to the case where  $\{\bar{T}, T^*, H\}$  is not necessarily irreducible.

## 6 Application to Construction of a Weyl representation

In the previous paper [8], a general structure was found to construct a Weyl representation from a weak Weyl representation. Here we recall it.

**Theorem 6.1** [8, Corollary 2.6] *Let  $(T, H)$  be a weak Weyl representation on a Hilbert space  $\mathcal{H}$  with  $T$  closed. Then the operator*

$$L := \log |H| \quad (6.1)$$

*is well-defined, self-adjoint and the operator*

$$D := \frac{1}{2}(TH + \overline{HT}) \quad (6.2)$$

is a symmetric operator. Moreover, if  $D$  is essentially self-adjoint, then  $(\overline{D}, L)$  is a Weyl representation of the CCR and  $\sigma(|H|) = [0, \infty)$ .

To apply this theorem, we need a lemma.

**Lemma 6.2** [6] *Let  $a \in \mathbb{R}$  and*

$$d_a := -\frac{1}{2}(p_{a,+}q_{a,+} + \overline{q_{a,+}p_{a,+}}) \quad (6.3)$$

*acting in  $L^2(\mathbb{R}_a^+)$ . Then  $d_a$  is essentially self-adjoint if and only if  $a = 0$ .*

**Theorem 6.3** *Let  $\mathcal{H}$  be separable and  $(T, H) \in W_+(\mathcal{H})$  with  $\inf \sigma(H) = 0$  and  $T$  closed. Suppose that  $\{T, T^*, H\}$  is irreducible. Let  $L$  and  $D$  be as in (6.1) and (6.2) respectively. Then  $D$  is essentially self-adjoint and  $(\overline{D}, L)$  is a Weyl representation of the CCR.*

*Proof.* Let  $\hat{d}_0$  be the operator  $d_0$  with  $p_{0,+}$  replaced by  $\overline{p_{0,+}}$ . Then, by Theorem 4.1,  $D$  is unitarily equivalent to  $\hat{d}_0$ . We have  $d_0 \subset \hat{d}_0$ . By Lemma 6.2,  $d_0$  is essentially self-adjoint. Hence  $\hat{d}_0$  is essentially self-adjoint. Therefore it follows that  $D$  is essentially self-adjoint. The second half of the theorem follows from Theorem 6.1. ■

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